2023-24 MATH2048: Honours Linear Algebra II

Homework 6

Due: 2023-10-30 (Monday) 23:59

For the following homework questions, please give reasons in your solutions. Scan your solutions and submit it via the Blackboard system before due date.

1. Let $V = P_1(\mathbb{R})$ and $W = \mathbb{R}^2$ with respective standard ordered bases β and γ. Define $T : V \to W$ by

$$T(p(x)) = (p(0) - 2p(1), p(0) + p'(0)),$$

where $p'(x)$ is the derivative of $p(x)$.

   (a) For $f \in W^*$ defined by $f(a, b) = a - 2b$, compute $T^*(f)$.

   (b) Compute $[T]_\beta^\gamma$ and $[T^*]_{\gamma^*}^{\beta^*}$ independently.

2. Let $V = P_n(F)$, and let $c_0, c_1, \ldots, c_n$ be distinct scalars in F.

   (a) For $0 \leq i \leq n$, define $f_i \in V^*$ by $f_i(p(x)) = p(c_i)$. Prove that $\{f_0, f_1, \ldots, f_n\}$ is a basis for V^*.

   (b) Show that there exist unique polynomials $p_0(x), p_1(x), \ldots, p_n(x)$ such that $p_i(c_j) = \delta_{ij}$ for $0 \leq i \leq n$. (Hint: Lagrange Polynomials)

   (c) For any scalars $a_0, a_1, \ldots, a_n$ (not necessarily distinct), find the polynomial $q(x)$ of degree at most n such that $q(c_i) = a_i$ for $0 \leq i \leq n$ and show that $q(x)$ is unique.

3. Let $A, B \in M_{n\times n}(\mathbb{C})$.

   (a) Prove that if B is invertible, then there exists a scalar $c \in \mathbb{C}$ such that $A + cB$ is not invertible. Hint: Examine $\det(A + cB)$.

   (b) Find nonzero 2×2 matrices A and B such that both A and $A+cB$ are invertible for all $c \in \mathbb{C}$.

4. (a) Let T be a linear operator on a vector space V over the field F, and let $g(t)$ be a polynomial with coefficients from F. Prove that if x is an eigenvector of T with corresponding eigenvalue λ, then $g(T)(x) = g(\lambda)x$. That is, x is an eigenvector of $g(T)$ with corresponding eigenvalue $g(\lambda)$.

   (b) Use (a) to prove that if $f(t)$ is the characteristic polynomial of a diagonalizable linear operator T, then $f(T) = T_0$, the zero operator. (Remark: This result does not depend on the diagonalizability of T.)

5. Let $A \in M_{n\times n}(F)$. Recall from §5.1 Q14 that A and A^t have the same characteristic polynomial and hence share the same eigenvalues with the same multiplicities. For any eigenvalue λ of A and A^t, let E_λ and E'_λ denote the corresponding eigenspaces for A and A^t, respectively.

   (a) Show by way of example that for a given common eigenvalue, these two eigenspaces need not be the same.

   (b) Prove that for any eigenvalue λ, $\dim(E_\lambda) = \dim(E'_\lambda)$.

   (c) Prove that if A is diagonalizable, then A^t is also diagonalizable.

The following are extra recommended exercises not included in homework.

1. Let V and W be finite-dimensional vector spaces over F. Let $\psi_1 : V \to V^{**}$ be defined by $\psi_1(v)(f) = f(v)$ for all $f \in V^*$ and $\psi_2 : W \to W^{**}$ be defined by $\psi_1(w)(g) = g(w)$ for all $g \in W^*$. Note that ψ_1 and ψ_2 are isomorphisms.

   Let $T : V \to W$ be linear, and define $T^{**} = (T^*)^*$. Prove that $\psi_2 T = T^{**}\psi_1$.

2. Let V and W be nonzero vector spaces over the same field, and let $T : V \to W$ be a linear transformation.

   (a) Prove that T is onto if and only if T^* is one-to-one.

   (b) Prove that T^* is onto if and only if T is one-to-one.

   Hint: Parts of the proof require the result of §2.6 Q19 for the infinite dimensional case.

3. Let A be an $n \times n$ matrix with characteristic polynomial

$$f(t) = (-1)^n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$$

(a) Prove that $f(0) = a_0 = \det(A)$. Deduce that A is invertible if and only if $a_0 \neq 0$.

(b) Prove that $f(t) = (A_{11}-t)(A_{22}-t)\cdots(A_{nn}-t)+q(t)$, where $q(t)$ is a polynomial of degree at most $n-2$. (Hint: Apply mathematical induction to n.)

(c) Show that $tr(A) = (-1)^{n-1}a_{n-1}$.